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An incremental variational approach and computational homogenization for elasto-damageable heterogeneous materials

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Résumé — We investigate the nonlinear effective behavior of quasi brittle composites with evolving damage. To this end, we rely on an incremental variational approach for which we first introduce a linearization of the elastic-damageable constitutive law. The resulting Linear Comparison Composite (LCC) is then homogenized by using classical Hashin-Shtrikman bounds. We provide a theoretical and numerical assessment of the incremental variational procedure applied to a porous materials. For particulate composites with a damageable matrix, a complete validation by means of full-field simulations are available in [4] and will be shown during the oral presentation.
Mots clés — Nonlinear Homogenization, Incremental variational approach, Damage, Heterogeneous materials, Effective behavior.

1 Introduction

Composite materials with a brittle matrix such as Ceramics Matrix Composites (CMC), or concrete materials are generally considered for various thermostructural applications in different industrial fields such as aeronautics, civil engineering, mechanical engineering, shipbuilding industry, etc. This class of composites allows to have satisfactory rigidity and to sustain high levels of loadings. However, a possible major limitation is their susceptibility to failure due to progressive and complex damage phenomena in the matrix. Damage modeling in brittle matrix composites can be efficiently addressed through the use of mean-field homogenization methods since they allow to obtain semi-analytical estimates of the nonlinear effective behavior which take account of the microstructural characteristics. Since the nineties, significant progress has been notably achieved in homogenization of composites whose nonlinear constituents' behaviors are described by means of a single potential. In the context of viscoelastic and/or elasto(visco)plastic composites where the coupling between elasticity and dissipative phenomena is of paramount importance, Lahellec and Suquet [3] have introduced an incremental variational approach dedicated to composites whose constituents' behaviors are described by means of two potentials (a free energy and a dissipation potential), as required for Generalized Standard Materials. This approach, known as the EIV one, has been developed and applied to viscoelastic composites and to elasto(visco)plastic composites by several authors among which [1, 2], etc.
Although significant advances have been achieved in the above-mentioned studies, taking into account damage phenomena in a fully coupled homogenization approach still remains a largely open and barely explored issue in solid mechanics. The very few works dealing with this topic are recent and at an early stage of development (see for instance [7]). Particular mention has to be made of the publication [5] in which the authors provide an approach that suitably combines analytical methods with theoretical and numerical homogenization of composites experiencing brittle damage. This first attempt of modeling, dedicated to a material involved in the nuclear industry, is in the same vein as our study which aims at developing a homogenization framework for appropriate effective laws of heterogeneous materials with elastic damageable constituents. Our concern will be primarily to investigate the responses of composites with those types of constituents. More precisely, the study relies for its theoretical part on the incremental variational procedure (see [3]). For simplicity we will consider an isotropic damage law within the framework of Generalized Standard Materials, i.e. based on a scalar damage variable and on the consideration of two potentials, namely a free energy and a dissipation potential.

2 An incremental variational model for elasto-damageable composites

2.1 The local damage model in the context of composites

Let us consider now a Representative Volume Element (RVE) Ω of a composite material consisting of N-phases with $\Omega^{(r)}$ the volume of each phase : $r(r=1,\ldots,N)$, the phases being assumed to have elasto-damageable behavior as modeled in the Generalized Standard Materials (GSM) framework. Thus, the mechanical behavior model of these phases can be formulated by considering the state variables deformation $\boldsymbol{\varepsilon}$ and an internal variable d (chosen as a positive scalar, so as to consider only isotropic damage processes) and by choosing two convex potentials that we will specify later.
The following characteristic function is used to specify the phase distribution :

$$\chi^{(r)}(x) = \begin{cases} 1 \text{ if } x \in \Omega^{(r)} \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

Reversible effects are associated with a free-energy density that depends only on the state variables of the material, $\boldsymbol{\varepsilon}$ the strain one and d the damage one :

$$\begin{cases} w(\boldsymbol{\varepsilon},d) = \sum_{r=1}^{N} w^{(r)}(\boldsymbol{\varepsilon},d)\,\chi^{(r)}(x) \quad \text{with :} \\ w^{(r)}(\boldsymbol{\varepsilon},d) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^{(r)}(d) : \boldsymbol{\varepsilon} \end{cases} \tag{2}$$

where $\mathbb{C}^{(r)}(d)$ represents the elasticity tensor of the damaged material.
For simplicity, we consider for the damage the following degradation function widely used in literature :

$$\mathbb{C}^{(r)}(d) = g(d)\,\mathbb{C}_s^{(r)} = (1-d)^2\,\mathbb{C}_s^{(r)} \tag{3}$$

where $\mathbb{C}_s^{(r)}$ represents the stiffness tensor of the undamaged material.
We denote $\varphi(\dot{d},d)$ and $\varphi^{(r)}(\dot{d},d)$ the dissipation potential over the composite and a phase r, respectively :

$$\begin{cases} \varphi(\dot{d},d) = \sum_{r=1}^{N} \varphi^{(r)}(\dot{d},d)\chi^{(r)}(x) \quad \text{with :} \\ \varphi^{(r)}(\dot{d},d) = \mathcal{Y}_c\,\dot{d} + \Psi_c(\dot{d}) \end{cases} \tag{4}$$

with $\Psi_c(\dot{d}) : \{\dot{d} \geq 0\}$ the irreversibility condition and $\mathcal{Y}_c$ is a characteristic of the damageable material.

2.2 Incremental variational principle

Methodology

In order to determine the effective condensed incremental potential $\tilde{w}_\Delta$, we apply the incremental variational procedure proposed by Lahellec and Suquet (2007). We recall the variational problem :

$$\begin{cases} w_\Delta(x,\boldsymbol{\varepsilon}) = \inf_d J(x,\boldsymbol{\varepsilon},d) \quad \text{with :} \\ J(x,\boldsymbol{\varepsilon},d) = \sum_{r=1}^{N} J^{(r)}(x,\boldsymbol{\varepsilon},d)\chi^{(r)}(x) \quad \text{and :} \\ J^{(r)}(x,\boldsymbol{\varepsilon},d) = w^{(r)}(\boldsymbol{\varepsilon},d) + \Delta t\varphi^{(r)}\left(\frac{d-d_n}{\Delta t}\right) \end{cases} \tag{5}$$

Following [3], the local behavior is linearized while taking into account heterogeneity of the incremental potential within the phases. Adressing these two steps separately as in [1, 2], could be a subject of future investigation. In our case, due to elasticity and damage coupling, the incremental potential J is approximated by a linearized incremental potential J_0 chosen in the form :

$$\begin{cases} J_0(x,\boldsymbol{\varepsilon},d) = \sum_{r=1}^{N} J_0^{(r)}(\boldsymbol{\varepsilon},d)\chi^{(r)}(x) \\ J_0^{(r)} = \frac{1}{2}(1-d)^2\mathcal{A}_0^{(r)} + \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}_0^{(r)} : \boldsymbol{\varepsilon} + \mathcal{Y}_c\,(d-d_n) + \Delta t\,\Psi_c\left(\frac{d-d_n}{\Delta t}\right) \end{cases} \tag{6}$$

where $\mathcal{A}_0^{(r)}$ is a uniform per phase scalar and $\mathbb{C}_0^{(r)}$ is a fourth-order tensor having the symmetries of an elasticity tensor, also uniform per phase.
The approximation of the condensed incremental potential is done by adding and subtracting to the potential J the linearized incremental potential J_0 *i.e.* $J = J_0 + J - J_0$, so that, on the one hand, the first term J_0 can be homogenized using classical homogenization schemes, and on the other hand, the difference $J - J_0$ can be estimated semi-analytically. The difference $\Delta J = J - J_0$ is then written

$$\begin{cases} \Delta J(x, \boldsymbol{\varepsilon}, d) = \sum_{r=1}^{N} \Delta J^{(r)}(\boldsymbol{\varepsilon}, d) \chi^{(r)}(x) \\ \Delta J^{(r)} = \frac{1}{2}(1-d)^2 \left[\boldsymbol{\varepsilon} : \mathbb{C}_s^{(r)} : \boldsymbol{\varepsilon} - \mathcal{A}_0^{(r)}\right] - \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}_0^{(r)} : \boldsymbol{\varepsilon} \end{cases} \tag{7}$$

By replacing J with its expression in the variational problem, the effective condensed incremental potential is expressed as

$$\tilde{w}_\Delta(\boldsymbol{E}) = \inf_{\boldsymbol{\varepsilon}/\langle\boldsymbol{\varepsilon}\rangle = E} \left[\inf_{d/\Psi_c(\dot{d})} \langle J_0(\boldsymbol{\varepsilon}, d) + \Delta J(\boldsymbol{\varepsilon}, d)\rangle \right] \tag{8}$$

the expressions of $J_0(\boldsymbol{\varepsilon}, d)$ and $\Delta J(\boldsymbol{\varepsilon}, d)$ to be considered being given in (6) and (7) respectively. As in [3], a upper bound on $\tilde{w}_\Delta$ can be obtained by taking a supremum condition of ΔJ with respect to $(\boldsymbol{\varepsilon}, d)$:

$$\tilde{w}_\Delta(\boldsymbol{E}) \leq \inf_{\boldsymbol{\varepsilon}/\langle\boldsymbol{\varepsilon}\rangle = E} \left[\inf_{d/\Psi_c(d)} \langle J_0(\boldsymbol{\varepsilon}, d)\rangle + \sup_{\boldsymbol{\varepsilon}^*, d^*} \langle \Delta J(\boldsymbol{\varepsilon}, d)\rangle \right] \tag{9}$$

Moreover, it has been shown that this upper bound may be too stiff in some cases. A way of relaxing this upper bound is usually to replace the supremum condition with a stationarity condition (see for instance [6]). Finally, the optimization with respect to the parameters introduced in J_0 provides an estimate of $\tilde{w}_\Delta$:

$$\tilde{w}_\Delta(\boldsymbol{E}) \approx \operatorname*{stat}_{\mathcal{A}_0^{(r)}, \mathbb{C}_0^{(r)}} \left[\inf_{\boldsymbol{\varepsilon}/\langle\boldsymbol{\varepsilon}\rangle = E} \left(\inf_{d/\Psi_c(\dot{d})} \langle J_0(\boldsymbol{\varepsilon}, d)\rangle + \operatorname*{stat}_{\boldsymbol{\varepsilon}^*, d^*} \langle \Delta J(\boldsymbol{\varepsilon}, d)\rangle \right) \right] \tag{10}$$

Note : In this procedure, the internal variable d will be approximated by solving the infimum problem $\inf_{d/\Psi_c(\dot{d})} \langle J_0(\boldsymbol{\varepsilon}, d)\rangle$. The reference variables, introduced in the linearization procedure, will be approximated by solving the stationarity problem $\operatorname*{stat}_{\boldsymbol{\varepsilon}^*, d^*} \langle \Delta J(\boldsymbol{\varepsilon}, d)\rangle$. The added $(.)^*$ aims at distinguishing the solution minimizing the first infimum problem from the one that is the solution to the stationarity problem.

Stationarity conditions

We now propose to develop all the stationarity conditions appearing in (10). We will first write the stationarity of ΔJ with respect to d^* and $\boldsymbol{\varepsilon}^*$ which will provide an expression for $\mathcal{A}_0^{(r)}$ and $\mathbb{C}_0^{(r)}$. The stationarity with respect to these two quantities will then allow to complete their expressions. Finally, the stationarity of J_0 with respect to d will provide an expression for damage as a function of $\boldsymbol{\varepsilon}$.

1. Stationarity of ΔJ

The stationarity of ΔJ (7) with respect to $\boldsymbol{\varepsilon}^*$ implies

$$\mathbb{C}_0^{(r)} = (1 - d^*)^2\, \mathbb{C}_s^{(r)} \tag{11}$$

Similarly, the stationarity condition of ΔJ over d^* leads to :

$$\mathcal{A}_0^{(r)} = \boldsymbol{\varepsilon}^* : \mathbb{C}_s^{(r)} : \boldsymbol{\varepsilon}^* \tag{12}$$

The two relations (11) and (12) show that the two unknowns $\mathcal{A}_0^{(r)}$ and $\mathbb{C}_0^{(r)}$ depend on the variables $\boldsymbol{\varepsilon}^*$ and d^* which remain to be determined.

2. Minimization of $J_0^{(r)}$

We obtain a Linear Comparison Composite (LCC) by minimizing $J_0^{(r)}$ with respect to d. And due to the stationarity of ΔJ with respect to the strain tensor, we can then write :

$$\tilde{w}_\Delta(\boldsymbol{E}) \approx \underset{\mathcal{A}_0^{(r)},\mathbb{C}_0^{(r)}}{\operatorname{stat}} \left[\inf_{\boldsymbol{\varepsilon}/\langle\boldsymbol{\varepsilon}\rangle=E} \langle w_0(\boldsymbol{\varepsilon})\rangle + \underset{\boldsymbol{\varepsilon}^*,d^*}{\operatorname{stat}} \langle \Delta J(\boldsymbol{\varepsilon},d)\rangle \right] \tag{13}$$

where $w_0(\boldsymbol{\varepsilon})$ is the energy associated with the resulting LCC such that :

$$w_0(\boldsymbol{\varepsilon}) = \inf_{d/\Psi_c(d)} J_0(\boldsymbol{\varepsilon},d) \tag{14}$$

The infimum of $J_0(\boldsymbol{\varepsilon},d)$ with respect to d is given with account of the irreversibility constraint on damage, which is written using the Karush-Kuhn-Tucker (KKT) optimality conditions, over each phase as :

$$\begin{cases} \dfrac{\partial}{\partial d}\left\{ \dfrac{1}{2}(1-d)^2 \mathcal{A}_0^{(r)} + \dfrac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}_0^{(r)} : \boldsymbol{\varepsilon} + \mathcal{Y}_c\,(d-d_n) \right\} + \lambda = 0 \\ \lambda \left(\dfrac{d-d_n}{\Delta t} \right) = 0 \\ \lambda \leq 0 \quad ; \dfrac{d-d_n}{\Delta t} \geq 0 \end{cases} \tag{15}$$

Case 1. $\lambda < 0$. According to $(15)_2$, this assumption leads to $d = d_n$ which is the same as the elastic case.
Case 2. $\lambda = 0$. It corresponds to $d - d_n > 0$, and thus to an evolving damage. From $(15)_1$, one gets :

$$\frac{\partial}{\partial d}\left\{ \frac{1}{2}(1-d)^2 \mathcal{A}_0^{(r)} + \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}_0^{(r)} : \boldsymbol{\varepsilon} + \mathcal{Y}_c\,(d-d_n) \right\} = 0 \tag{16}$$

The resulting expression of the damage variable is as follows :

$$d_{opt}^{(r)} = 1 - \frac{\mathcal{Y}_c}{\mathcal{A}_0^{(r)}} \tag{17}$$

which turns out to be uniform per phase. Despite this uniformity, the optimal damage field $d_{opt}^{(r)}$ will be shown to depend on the strain heterogeneity in phase r via its link with $\mathcal{A}_0^{(r)}$.

The energy associated with the LCC (see its definition by (14), with J_0 given by (6)) reads then :

$$w_0^{(r)}(\boldsymbol{\varepsilon}) = \frac{1}{2}\left(1-d_{opt}^{(r)}\right)^2 \mathcal{A}_0^{(r)} + \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}_0^{(r)} : \boldsymbol{\varepsilon} + \mathcal{Y}_c\left(d_{opt}^{(r)} - d_n\right) \tag{18}$$

The resulting $w_0^{(r)}$ is the LCC's linear thermoelastic potential with piecewise uniform parameters and, unlike the EIV formulation in [3], this potential does not involve a polarization $\tau_0^{(r)}$.
The effective energy of the LCC, (see (13)), is given by the minimization condition with respect to $\boldsymbol{\varepsilon}$:

$$\tilde{w}_\Delta(\boldsymbol{E}) \approx \underset{\mathcal{A}_0^{(r)},\mathbb{C}_0^{(r)}}{\operatorname{stat}} \left[\tilde{w}_0(E) + \underset{\boldsymbol{\varepsilon}^*,d^*}{\operatorname{stat}} \langle \Delta J(\boldsymbol{\varepsilon},d)\rangle \right] \tag{19}$$

where the effective energy of the LCC is

$$\tilde{w}_0(E) = \inf_{\boldsymbol{\varepsilon}/\langle\boldsymbol{\varepsilon}\rangle=E} \langle w_0(\boldsymbol{\varepsilon})\rangle \tag{20}$$

We are ready now to write the last optimality conditions, which will allow to determine the expressions of the remaining quantities $\mathcal{A}_0^{(r)}$ and $\mathbb{C}_0^{(r)}$.

3. Stationarity of $\left\langle J_0^{(r)} + \Delta J^{(r)} \right\rangle^{(r)}$

Optimality with respect to $\mathcal{A}_0^{(r)}$ and $\mathbb{C}_0^{(r)}$ allows to obtain the missing link between $(\boldsymbol{\varepsilon}^*, d^*)$ and $(\boldsymbol{\varepsilon}, d)$:

$$\underset{\mathcal{A}_0^{(r)}}{\text{stat}} \left\langle J_0^{(r)} + \Delta J^{(r)} \right\rangle^{(r)} \Longrightarrow \left\langle (1-d)^2 \right\rangle^{(r)} = \left\langle (1-d^*)^2 \right\rangle^{(r)} \tag{21}$$

$$\underset{\mathbb{C}_0^{(r)}}{\text{stat}} \left\langle J_0^{(r)} + \Delta J^{(r)} \right\rangle^{(r)} \Longrightarrow \begin{cases} \langle \boldsymbol{\varepsilon} : \mathbb{K} : \boldsymbol{\varepsilon} \rangle^{(r)} = \langle \boldsymbol{\varepsilon}^* : \mathbb{K} : \boldsymbol{\varepsilon}^* \rangle^{(r)} \\ \\ \langle \boldsymbol{\varepsilon} : \mathbb{J} : \boldsymbol{\varepsilon} \rangle^{(r)} = \langle \boldsymbol{\varepsilon}^* : \mathbb{J} : \boldsymbol{\varepsilon}^* \rangle^{(r)} \end{cases} \tag{22}$$

for which $\mathbb{K}$ represents the isotropic deviatoric projector and $\mathbb{J}$ the isotropic spherical projector.
By denoting $\langle a \rangle^{(r)}$ the average of the quantity a considered on the phase (r), from (12) and (22), it is readily seen that $\mathcal{A}_0^{(r)}$ can then be determined using the second-order moment estimate of the strain field :

$$\mathcal{A}_0^{(r)} = \left\langle \boldsymbol{\varepsilon} : \mathbb{C}_s^{(r)} : \boldsymbol{\varepsilon} \right\rangle^{(r)} \tag{23}$$

which is the average over the r phase of the elastic energy of the undamaged material.
Using the result (23), one can rewrite (17) in the following form

$$\left(1 - d_{opt}^{(r)}\right) \left\langle \boldsymbol{\varepsilon} : \mathbb{C}_s^{(r)} : \boldsymbol{\varepsilon} \right\rangle^{(r)} - \mathcal{Y}_c = 0 \tag{24}$$

which provides the optimal damage value :

$$d_{opt}^{(r)} = 1 - \frac{\mathcal{Y}_c}{\left\langle \boldsymbol{\varepsilon} : \mathbb{C}_s^{(r)} : \boldsymbol{\varepsilon} \right\rangle^{(r)}} \tag{25}$$

and then $\mathbb{C}_0^{(r)} = \left(1 - d_{opt}^{(r)}\right)^2 \mathbb{C}_s^{(r)}$ which represents the elasticity tensor of the Linear Comparison Composite (LCC) described by $w_0(\boldsymbol{\varepsilon})$ (18).

Effective response of the elasto-damageable composite

Once all the parameters are known, the effective behavior of the composite can be specified. In order to do so, we proceed to the minimization of $\tilde{w}_\Delta$ defined by (10), with respect to $\boldsymbol{\varepsilon}$. This functional is stationary with respect to $\mathcal{A}_0^{(r)}$, $\mathbb{C}_0^{(r)}$ and d, and the term ΔJ of this functional is stationary with respect to $\boldsymbol{\varepsilon}^*$ and d^*. Given this, we can define the local problem that corresponds to the Euler-Lagrange equations providing the solution to the variational problem :

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}(x) = 0 & \forall x \in \Omega \\ \\ \boldsymbol{\sigma}(x) = \mathbb{C}_0^{(r)} : \boldsymbol{\varepsilon}(x) & \forall x \in \Omega \\ \\ \langle \boldsymbol{\varepsilon}(x) \rangle = \boldsymbol{E} \quad + & \text{Boundary conditions on } \partial\Omega \end{cases} \tag{26}$$

The effective behavior of the nonlinear composite can then be estimated as :

$$\begin{aligned} \boldsymbol{\Sigma} &= \langle \boldsymbol{\sigma}(x) \rangle = \frac{\partial \tilde{w}_\Delta}{\partial \boldsymbol{E}}(\boldsymbol{E}) \approx \frac{\partial \tilde{w}_0}{\partial \boldsymbol{E}}(\boldsymbol{E}) \\ &= \sum_{r=1}^{N} c^{(r)} \, \mathbb{C}_0^{(r)} : \langle \boldsymbol{\varepsilon}(x) \rangle = \sum_{r=1}^{N} c^{(r)} \, \mathbb{C}_0^{(r)} : \left[\mathbb{A}^{(r)} : \boldsymbol{E} \right] \end{aligned} \tag{27}$$

where $\mathbb{A}^{(r)}$ is a fourth-order tensor that corresponds to the strain localization tensor in the phase (r). And $c^{(r)}$ is the volume fraction of the phase (r) such that : $c^{(r)} = \dfrac{|\Omega^{(r)}|}{|\Omega|}$.

Case of isotropic elasto-damageable matrix reinforced by elastic particles

The above-developed procedure is now particularized to the case of a two-phase particulate composite (N=2 being the number of phases), composed of an isotropic elasto-damageable matrix reinforced by randomly and isotropically distributed elastic spherical particles. The subscripts (1) and (2) represent the inclusion and the matrix, respectively. In the case of isotropic elastic behavior of the phases, the elasticity stiffness of the sound matrix takes the form :

$$\mathbb{C}_s^{(2)} = 3\,k_s^{(2)}\,\mathbb{J} + 2\mu_s^{(2)}\,\mathbb{K} \tag{28}$$

where $k_s^{(2)}$ and $\mu_s^{(2)}$ are the bulk modulus and the shear modulus, respectively. Tensor $\mathbb{K} = \mathbb{I} - \mathbb{J}$ represents the deviatoric projector of isotropic fourth-order tensors having the symmetries of an elasticity tensor, while $\mathbb{I}$ is the symmetric fourth-order identity tensor and $\mathbb{J}$ the spherical projector whose expressions are respectively $\mathbb{I}_{ijkl} = \dfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$ and $\mathbb{J} = \dfrac{1}{3}\delta_{ij}\delta_{kl}$.

In order to estimate the effective behavior of the composite as well as the first and second-order moments of the different fields in each phase, we made use of the Hashin-Shtrikman bounds.

Regarding the first-order moment of the strain $\boldsymbol{\varepsilon}$ upon the phases, it can be expressed as :

$$\bar{\boldsymbol{\varepsilon}}^{(1)} = \mathbb{A}^{(1)} : E\,, \quad \bar{\boldsymbol{\varepsilon}}^{(2)} = \frac{1}{c^{(2)}}\left(\boldsymbol{E} - c^{(1)}\bar{\boldsymbol{\varepsilon}}^{(1)}\right) \tag{29}$$

For a two-phase composite (N=2), the strain localization tensors associated with the Hashin-Shtrikman estimates are classically given by :

$$\mathbb{A}^{(1)} = \left(\mathbb{I} + c^{(2)}\mathbb{P}_0^{(2)} : \Delta\mathbb{C}_0\right)^{-1}\,, \quad \mathbb{A}^{(2)} = \frac{1}{c^{(2)}}\left(\mathbb{I} - c^{(1)}\mathbb{A}^{(1)}\right) \tag{30}$$

where $\Delta\mathbb{C}_0 = \mathbb{C}_0^{(1)} - \mathbb{C}_0^{(2)}$, and $\mathbb{P}_0^{(2)}$ is Hill tensor associated with the matrix phase of the LCC in the case of isotropically distributed spherical particles. Under these conditions, the tensor $\mathbb{P}_0^{(2)}$ is written as

$$\begin{cases} \mathbb{P}_0^{(2)} = \dfrac{\alpha_0^{(2)}}{3\,k_0^{(2)}}\mathbb{J} + \dfrac{\beta_0^{(2)}}{2\mu_0^{(2)}}\mathbb{K}, \\ \text{with } \alpha_0^{(2)} = \dfrac{3\,k_0^{(2)}}{3k_0^{(2)} + 4\mu_0^{(2)}},\ \beta_0^{(2)} = \dfrac{6\left(k_0^{(2)} + +2\mu_0^{(2)}\right)}{5\left(3k_0^{(2)} + 4\mu_0^{(2)}\right)} \end{cases} \tag{31}$$

Given these expressions, in order to obtain the first-order moment of the strain tensor within each constitutive phase, we need to determine $\mathbb{C}_0^{(r)}$. The latter, as we saw in (25), can be obtained with the use of the second-order moment of the strain tensor, which is estimated with the following expression :

$$\langle \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} \rangle^{(r)} = \frac{1}{c^{(r)}}\frac{\partial \tilde{w}_0}{\partial \mathbb{C}_0^{(r)}} \tag{32}$$

In this two-phase case, the elasticity of the inclusion is given by : $\mathbb{C}_0^{(1)} = \mathbb{C}_s^{(1)}$ as it remains undamaged. In the matrix phase, the expression of $\mathbb{C}_0^{(2)}$ which characterizes the LCC w_0 is obtained using the solution of the stationarity condition detailed above.

The nonlinear problem is reduced to the solution of a system of equations defined by two coupled functions whose unknowns are $\mathcal{A}_0^{(r)}$ and $\mathbb{C}_0^{(2)}$:

$$\begin{cases} F_1\left(\mathcal{A}_0^{(2)}, \mathbb{C}_0^{(2)}\right) = \mathcal{A}_0^{(2)} - \left\langle \boldsymbol{\varepsilon} : \mathbb{C}_s^{(2)} : \boldsymbol{\varepsilon} \right\rangle^{(2)} = 0 \\ F_2\left(\mathcal{A}_0^{(2)}, \mathbb{C}_0^{(2)}\right) = \mathbb{C}_0^{(2)} - \left(\dfrac{\mathcal{Y}_c}{\left\langle \boldsymbol{\varepsilon} : \mathbb{C}_s^{(2)} : \boldsymbol{\varepsilon} \right\rangle^{(2)}}\right)^2 \mathbb{C}_s^{(2)} = 0 \end{cases} \tag{33}$$

The resolution of this system allows to determine the damage level and then the mechanical behavior.

2.3 Algorithmics

We now consider the numerical implementation of the developed theoretical model. The aim here is to describe the algorithm allowing to determine, for a given loading history defined in terms of prescribed macroscopic strain E at all times t_n, the effective stress $\boldsymbol{\Sigma}$ and the first and second-order moments of the local fields in the phases. This problem will be solved iteratively by determining the solution at t_{n+1} from the known solution at the previous time step t_n. The algorithm's structure goes as follows :

— At the current time step $t = t_{n+1}$, the macroscopic strain $\boldsymbol{E}_n$ and the field statistics at step t_n are known. An increment $\Delta\boldsymbol{E}$ is applied. At each step, we aim at determining the macroscopic stress $\boldsymbol{\Sigma}_{n+1}$ and the local field statistics (namely the first and second order moments) at time step t_{n+1}. The coupled nonlinear system (33) is therefore solved at t_{n+1}. To this end, the equation associated with F_2 of this system can be decomposed into spherical and deviatoric parts. These quantities are obtained through (29) and (32), the resulting expressions depend on $k_0^{(2)}$ and $\mu_0^{(2)}$.
Solving the resulting system provides the elastic characteristics of the LCC $\mathcal{A}_0^{(2)}$, $k_0^{(2)}$ and $\mu_0^{(2)}$. For this purpose, we use the function LEAST_SQUARES on Python which allows the nonlinear resolution of the system using the Levenberg-Marquardt algorithm, making it possible to determine the energy $w_0^{(r)}$ (18) of the LCC with phase-homogeneous properties as well as the values of the matrix damage (17).

— The second step consists in calculating the homogenized parameters $\left(\tilde{\mu}_0, \tilde{k}_0\right)$ defining $\tilde{w}_0$, by implementing a Hashin-Shtrikman type bound. Once all these parameters are determined, we can proceed to the calculation of the macroscopic stress $\boldsymbol{\Sigma}$ which is obtained using the equation (27).

— All the quantities are then updated at the end of the current time step, and used for the following time step $(.)_{n+1} \longrightarrow (.)_n$

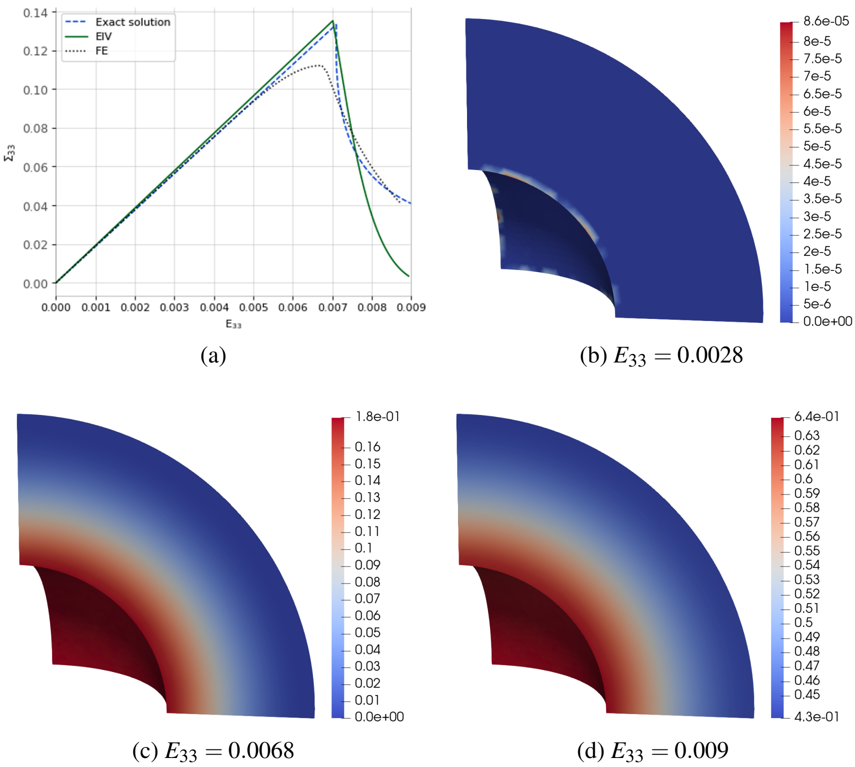
3 A simple example of validation of the variational homogenization model

In this section we consider, an elasto-damageable hollow sphere submitted to a uniform radial displacement. A finite element simulation of this problem is carried out, using a regularized gradient damage model whose detail can be found in [4]. The results are compared to closed-form solutions also available in [4] and to the predictions of the developped incremental variational homogenization approach.

Considering the symmetry of the problem, the numerical simulation can be conducted on one-eighth of the sphere. The sphere's dimensions ratio is $b/a_0 = 2$, $b = 1$mm is the external radius, and $a_0 = 0.5$mm is the internal one (which corresponds to a porosity $f = 0.125$). The material properties are also chosen as follows : the ratio of the bulk modulus and the shear modulus $k_s^{(2)}/\mu_s^{(2)} = 30$ (near an elastic incompressible sound matrix). The mesh size considered here amounts to $h \approx 0.016$mm, with the total number of elements reaching nearly 68000. The internal length considered in the regularized damage model of the matrix is equal to $l_0 = 0.1$mm. The threshold $\mathcal{Y}_c = \dfrac{3G_c}{8l_0} = 0.013$MPa. Results are shown on figure 1.

When it comes to the full-field simulations, a notable difference can be observed in the reached threshold before the softening takes place. Indeed, for the closed-form solution, there is the purely elastic phase until the threshold is reached and then the damaged phase starts. Whereas in the FEM simulations, it has been observed that damage begins to evolve at $E_{33} = 0.0025$, in other words before reaching the highest stress value. Schematically, it is clearly seen from the damage pattern evolution (see Fig. 1b-1d) in the numerical simulation that there is a competition between the damaged zone and a linear elastic one, the softening occurring only after a stage of damage development.

As mentioned before, Fig. 1b-1d represent the evolution of the damage field within the hollow sphere at three different levels of loading. The first one corresponds to the first occurrence of damage within the matrix, the second level corresponds to the maximum stress level (that is right before the softening regime), and the last one corresponds to the end of the simulation with a damage level ranging from 0.45 -the minimum value for d- to 1. Note that damage starts from the inner radius and progresses as expected in the sphere when the applied deformation increases. A good agreement is obtained, confirming then an interest in the theoretical results.



(a) (b) $E_{33} = 0.0028$

(c) $E_{33} = 0.0068$ (d) $E_{33} = 0.009$

FIGURE 1 – (a) Comparison of the mechanical responses of a hollow sphere submitted to a uniform radial displacement : analytical solution, the variational developed approach and full-field simulations using a regularized damage model. (b)-(c)-(d) : Damage field at different loading levels

4 Conclusion

In this study, we present an incremental variational approach for nonlinear effective behavior of composites with evolving damage. The approach has led to a semi-analytical formulation of the model which has been implemented numerically. By considering a porous material with an elastic damageable matrix, a first example of validation has been shown and allowed to illustrate the predictive capabilities of the model. More extensive validation for particulate composites by comparing the model predictions to full-field simulations is available and can be found in [4]. The latter also provides information on the fields' statistics including fluctuations in the constitutive phases. Further extensions of this study can be devoted to composites in which elastoplastic behaviors are coupled to damage.

Références

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