Approximation of frictional contact using Nitsche's methods in 3D elasto-plastic problems

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Abstract — We present recent developments in Nitsche-type methods for certain contact and friction problems. We recall the parameterization of the Nitsche method in the case of a unilateral contact with Coulomb friction in elastoplasticity. Some 3D industrial examples under SYSTUS/ SYSWELD are also shown.
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1 Introduction

Frictional, elasto-plastic multi-body contact problems play an important role in mechanical engineering. The non-linearities caused by geometric contact and frictional constraints, combined with the non-linearity in the material law, result in challenging numerical problems in the form of variational inequalities. For which, efficient solving methods are needed. Numerical methods for contact problems have been an active field of research for many years, yet new methods continue to emerge. Probably the youngest member of this research family is Nitsche's method, see [12], [11], [10]. Originally introduced for the weak imposition of boundary conditions, it has since been applied to various interface-coupled problems. The first application to contact mechanics was presented in [19], and a mathematical analysis for linearized kinematics has been published by F. Chouly, P. Hild, and Y. Renard in [4], [6], and [5]. Unlike other methods such as Lagrangian or penalty methods, Nitsche's method is simultaneously variationally consistent (and therefore optimally convergent) and does not introduce any additional degrees of freedom [9]. No Lagrange multiplier is needed, and no discrete inf-sup condition must be fulfilled, contrary to mixed methods. This comes at the expense of having to evaluate the boundary traction from the continuum stresses. Additionally, more advanced variants require an adjoint term, including the stresses computed from the test function, to obtain, depending on a parameter, either a symmetric variant or a skew-symmetric variant, which is stable for any positive penalty parameter, see [2], [3]. Very recently, the first application of Nitsche's method to finite deformation elasto-plastic contact problems has been presented in [18].

In this note, we describe the use of Nitsche's method to prescribe contact (with or without Coulomb friction condition) between two elasto-plastic bodies. This corresponds to a weak integral contact condition which has some similarities with the ones using Lagrange multipliers. The goal of this note is to present how different industrial cases in SYSTUS/SYSWELD have utilized Nitsche's method to solve some contact problems within the small deformations framework. The approximation strategy proposed here was first implemented in the open-source finite element library GetFEM [13] for small and large elastic or hyperelastic contact, with or without friction.

2 The unilateral contact problem : master-slave formulation

In this section, we present the strong and weak formulations of the studied frictional two-body contact problem with an elasto-plastic material law and linear isotropic hardening. We consider two deformable bodies Ω_α, $\alpha = 1$ or 2; these are domains with piecewise C^1 boundaries included in $\mathbb{R}^{ndim}$, where $ndim = 2$ or 3, representing the reference configurations of two elastic bodies. The boundary Γ_1 of Ω_1 (and Γ_2 of Ω_2 respectively) is divided into three non-overlapping parts: $\Gamma_{1,C}$, the slave potential zone of contact with $meas(\Gamma_{1,C}) > 0$ (and $\Gamma_{2,C}$, respectively, with $meas(\Gamma_{2,C}) > 0$); $\Gamma_{1,N}$, the Neumann part (and $\Gamma_{2,N}$ respectively), and $\Gamma_{1,D}$, the Dirichlet part with $meas(\Gamma_{1,D}) > 0$ (and $\Gamma_{2,D}$, respectively, with $meas(\Gamma_{2,D}) > 0$).

The two bodies are subjected to volume forces $\vec{\mathbf{f}} = (\vec{\mathbf{f}}_1, \vec{\mathbf{f}}_2)$ on $\Omega_1 \times \Omega_2$, to surface loads $\vec{\ell} = (\vec{\ell}_1, \vec{\ell}_2)$ on $\Gamma_{1,N} \times \Gamma_{2,N}$ and satisfy non-homogeneous boundary Dirichlet conditions on $\Gamma_{1,D} \times \Gamma_{2,D}$, with the displacement being prescribed to the given value $\vec{\mathbf{u}}_D = (\vec{\mathbf{u}}_{1,D}, \vec{\mathbf{u}}_{2,D})$.

Superscript 1 denotes the slave surface and superscript 2 denotes the master surface. $\vec{\mathbf{X}}$ is the position of a point on the slave surface $\Gamma_{1,C}$ and $\vec{\mathbf{Y}}$ is its closest point projection on the master surface $\Gamma_{2,C}$.

We are interested in the displacements $\vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2)$ and assume small (elasto-plastic) deformations for the two bodies.

Definition 2.1 *Let $\Omega^0_\alpha = \Omega_\alpha$ be the reference configuration of the deformable solids in a space of dimension $ndim = 2$ or 3. A deformed configuration Ω^t_α of the considered solids can be defined through a transformation known as* motion *or* deformation φ *which maps any point $\vec{\mathbf{X}}$ of the reference configuration to a point $\vec{\mathbf{x}}$ of the deformed one. We define the* displacement $\vec{\mathbf{u}}$ *relatively to the reference configuration as:*

$$\vec{\mathbf{u}}(\vec{\mathbf{X}}) := \vec{\varphi}(\vec{\mathbf{X}}) - \vec{\mathbf{X}} = (\vec{\varphi} - \boldsymbol{id})(\vec{\mathbf{X}}), \ \textit{with :} \ \vec{\mathbf{x}} := \vec{\varphi}(\vec{\mathbf{X}}).$$

The displacement is defined as the difference between the reference and the current configuration. The velocity *of a material point is the derivative of the motion with $\vec{\mathbf{X}}$ fixed :*

$$\vec{\mathbf{v}}(\vec{\mathbf{X}}, t) = \frac{\partial}{\partial t}(\vec{\varphi}(\vec{\mathbf{X}}, t)) \ .$$

In the deformed configuration Ω^t_α, at time t, different portions of the boundary $\partial\Omega^t_1$ of Ω^t_1 may come into contact and interact with Ω^t_2. A non-penetration condition on the deformed contact surfaces $\Gamma^t_{1,C}$ and $\Gamma^t_{2,C}$ can be expressed with the help of a mapping function linking a point $\vec{\mathbf{X}}$ to its mapping $\Pi(\vec{\mathbf{X}})$. We denote by $\Gamma^t_{\alpha,C} \subset \Gamma^t_\alpha$ (resp. $\Gamma^0_{\alpha,C} \subset \Gamma^0_\alpha$) the set of points $\vec{\mathbf{X}}$ (resp. $\vec{\mathbf{X}}$) in the deformed (resp. reference) configuration. Recall that points $\vec{\mathbf{X}}$ and $\vec{\mathbf{x}}$ have dimensions of ndim; $\vec{\mathbf{X}}$ represents Material or Lagrangian coordinates, while $\vec{\mathbf{x}}$ represents Spatial or Eulerian coordinates.

Definition 2.2 *The term* small perturbation hypothesis *(or even the small displacement hypothesis) comprises the assumptions of small displacements, infinitesimal transformations, and infinitesimal deformations, which enables us to proceed with the physical linearization of the constitutive law for the material. Then, the* linearized strain tensor field *is given by* $\varepsilon(\vec{\mathbf{u}}) = \frac{1}{2}(\nabla\vec{\mathbf{u}} + \nabla\vec{\mathbf{u}}^\top)$

The displacements $\vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2)$ fulfill the following conditions for $\alpha = 1, 2$:

$$\varepsilon(\vec{\mathbf{u}}_\alpha) = \mathbf{H}_\alpha \boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha) + \varepsilon_{\alpha,plas} \text{ in } \Omega_\alpha \tag{1}$$

Relation (1) describes the material law, relating the linearized strain $\varepsilon(\vec{\mathbf{u}}_\alpha)$ to the stress $\boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha)$. The strain is divided into an elastic part $\mathbf{H}_\alpha \boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha)$, where $\mathbf{H}_\alpha$ is the fourth-order symmetric elasticity tensor (compliance tensor corresponding to isotropic material that satisfies the usual uniform ellipticity and boundedness properties), and a plastic part $\varepsilon_{\alpha,plas}$.

Then, the stress tensor field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})_{1 \le i,j \le d = 2 \text{ or } 3}$ is given by

$$\boldsymbol{\sigma}(\vec{\mathbf{u}}) = \mathbf{H}^{-1}_\alpha(\varepsilon(\vec{\mathbf{u}}) - \varepsilon_{\alpha,plas}).$$

Consequently, the displacement $\vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2)$ on $\Omega_1 \times \Omega_2$ must satisfy the following set of equations, apart from the contact condition, which will be described later:

$$\left\{ \begin{array}{ccc} \text{Find } \vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2) \text{ satisfying} & & \\ -\mathrm{div}\boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha) = \vec{\mathbf{f}}_\alpha & \text{in} & \Omega_\alpha, \\ \boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha) = \mathbf{H}^{-1}_\alpha(\varepsilon(u_\alpha) - \varepsilon_{\alpha,plas}) & \text{in} & \Omega_\alpha, \\ \varepsilon_{\alpha,plas}(\boldsymbol{\tau} - \boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha)) \ge 0, \ \ \forall\tau \text{ with } \mathcal{F}_{\alpha,iso}(\tau, |\varepsilon_{\alpha,plas}|F) \le 0 & \text{in} & \Omega_\alpha, \\ \vec{\mathbf{u}}_\alpha = \vec{\mathbf{u}}_{\alpha,D} & \text{on} & \Gamma_{\alpha,D}, \\ \boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha)\vec{\mathbf{n}}_\alpha = \vec{\ell}_\alpha & \text{on} & \Gamma_{\alpha,N}. \end{array} \right. \tag{2}$$

The deviatoric part of a tensor τ is denoted by $\tau^{dev} := \tau - \frac{1}{d}tr(\tau)\mathbb{I}_{d\times d}$ and $|\cdot|_F$ denotes the Frobenius norm. The yield function $\mathcal{F}_{\alpha,iso}$ is defined as $\mathcal{F}_{\alpha,iso} = |\tau^{dev}|_F - (\sigma^0_\alpha + \xi_{\alpha,iso}\eta)$, where σ^0_α is the yield stress and $\xi_{\alpha,iso}$ is the isotropic hardening parameter. It is important to note that the complementarity condition ensures that plastic strain may only occur if the yield function equals zero.

Now, concerning the contact conditions, let us introduce some important definitions.

Definition 2.3 Π *denotes the* orthogonal projection *from the slave boundary* $\Gamma_{1,C}$ *onto the master boundary* $\Gamma_{2,C}$*:*

$$\Pi : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \Gamma_{2,C} \\ \vec{\mathbf{X}} & \mapsto & \Pi(\vec{\mathbf{X}}) = \vec{\mathbf{Y}}. \end{array} \tag{3}$$

Remark 2.1 *The operator* Π *is assumed to be a* C^1 *one-to-one correspondence on* $\Pi(\Gamma_{1,C})$ *(this hypothesis is satisfied, for instance, when* $\Gamma_{\alpha,C}$ *is convex and* C^1 *for* $\alpha \in \{1,2\}$*).*

Definition 2.4 *The* outward unit normal vector *on* $\Gamma_{1,C}$ *is denoted by* $\vec{\mathbf{n}}_1$ *or* $\vec{\mathbf{n}}_X$.

The outward unit normal vector *on* $\Gamma_{2,C}$ *is denoted by* $\vec{\mathbf{n}}_2$.

The outward unit normal vector *for the contact condition, denoted by* $\vec{\mathbf{n}}_Y$*, is chosen to be that of* $\Gamma_{2,C}$*:*

$$_Y : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \mathbb{R}^d \\ \vec{\mathbf{X}} & \mapsto & n_2(\Pi(\vec{\mathbf{X}})). \end{array}$$

The orthonormal basis, denoted by $(\vec{\mathbf{t}}_1, \vec{\mathbf{t}}_2)$*, is the contravariant tangential basis vector defined at the point* $\vec{\mathbf{Y}} = \Pi(\vec{\mathbf{X}})$ *on the master surface.*

Definition 2.5 *The* initial gap g^0 *between* $\Gamma_{1,C}$ *and* $\Gamma_{2,C}$ *is defined as the distance function:*

$$g^0 : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \mathbb{R} \\ \vec{\mathbf{X}} & \mapsto & (\vec{\mathbf{X}}_1 - \Pi(\vec{\mathbf{X}}_1)) \cdot \vec{\mathbf{n}}_Y. \end{array} \tag{4}$$

Remark 2.2 *The gap function, corresponding to ray-tracing with respect to a point x, is defined by:*

$$g : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \mathbb{R} \\ \vec{\mathbf{X}} & \mapsto & (\vec{\mathbf{X}}_1 - \Pi(\vec{\mathbf{X}}_1)) \cdot \vec{\mathbf{n}}_1. \end{array} \tag{5}$$

Definition 2.6 *For a displacement field* $\vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2)$ *defined on* $\Omega_1 \times \Omega_2$*, the* normal jump *on the slave boundary* Γ_1 *for the normal displacement is defined as follows:* $[\![\cdot\vec{\mathbf{n}}]\!] = (\vec{\mathbf{u}}_2 \circ \Pi - \vec{\mathbf{u}}_1) \cdot \vec{\mathbf{n}}$.

Concerning the normal stress, we define

$$\boldsymbol{\sigma}(\vec{\mathbf{u}}_1)\vec{\mathbf{n}}_1 = -\sigma_n(\vec{\mathbf{u}}_1)\vec{\mathbf{n}}_1 + \boldsymbol{\sigma}_t(\vec{\mathbf{u}}_1) \qquad \text{with } \sigma_n(\vec{\mathbf{u}}_1) = \sigma(\vec{\mathbf{u}}_1)\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}$$

where $\vec{\mathbf{n}}_1$ is the unit normal vector defined at point $\vec{\mathbf{X}}$ (in the discretized configuration, its orientation dependes on the orientation of the node numbering for each element), and $\boldsymbol{\sigma}_t(\vec{\mathbf{u}}_1)$ is the frictional traction applied to the master surface by the point $\vec{\mathbf{X}}$ on the slave surface.

Furthermore,

$$\sigma(\vec{\mathbf{u}}_2 \circ \Pi)n_2 \circ \Pi = \sigma_n(\vec{\mathbf{u}}_2 \circ \Pi)\vec{\mathbf{n}} + \sigma_t(\vec{\mathbf{u}}_2 \circ \Pi) \qquad \text{with } \sigma_n(\vec{\mathbf{u}}_2 \circ \Pi) = \sigma(\vec{\mathbf{u}}_2 \circ \Pi)n_2 \circ \Pi \cdot \vec{\mathbf{n}}.$$

Definition 2.7 *This allows us to define the* normal stress jump *as* $[\![\sigma(\vec{\mathbf{u}})\vec{\mathbf{n}}]\!] = \sigma(\vec{\mathbf{u}}_1)n_1 + \sigma(\vec{\mathbf{u}}_2 \circ \Pi)n_2 \circ \Pi\ |det(J_\Pi)|$*, with* J_Π *denoting the Jacobian matrix of* Π.

With these jumps defined, the unilateral frictional contact conditions can be expressed on the slave boundary $\Gamma_{1,C}$ as follows:

$$
\begin{cases}
[\![\vec{\mathbf{u}}\cdot\vec{\mathbf{n}}]\!] \le g & (i),\\
\sigma_n(\vec{\mathbf{u}}_1) \le 0 & (ii),\\
\sigma_n(\vec{\mathbf{u}}_1)([\![\vec{\mathbf{u}}\cdot\vec{\mathbf{n}}]\!]-g)=0 & (iii),\\
[\![\sigma(\vec{\mathbf{u}})\vec{\mathbf{n}}]\!]=0 & (iv),\\
\|\sigma_t(\vec{\mathbf{u}})\| \le -F\sigma_n(\vec{\mathbf{u}}) \ \text{ if } \ d_t=0, & (v).\\
\sigma_{\vec{\mathbf{t}}}(\vec{\mathbf{u}}) = F\sigma_n(\vec{\mathbf{u}})\dfrac{d_t}{\|d_t\|} \ \text{ if } \ d_t\neq 0 & (vi).
\end{cases} \tag{6}
$$

Equations (6)(v) and (vi) represent the Coulomb friction conditions. To formulate these friction conditions, a coefficient of friction, denoted by $F \geq 0$, is necessary, along with a rigorous notion of sliding velocity. However, in a quasi-static evolution context, instead of a sliding velocity, we use a tangential displacement increment, denoted by d_t. Duvaut and Lions (1972) describe the expression $d_t(u_1,u_2) = (I-n\otimes n)[\![u]\!]$, which, despite being somewhat artificial, exhibits the same characteristics as those obtained for an expression of d_t derived from a time discretization, expressed as:

$$d_t(u_1,u_2) = (I-n\otimes n)([\![u]\!]-[\![u^0]\!])$$

where $[\![u^0]\!]$ is the displacement jump at the previous time step.

Now, let us introduce the Hilbert space V and the convex cone K of admissible displacements:

$$V := H^1(\Omega_1)^d \times H^1(\Omega_2)^d,$$

$$K := \{\delta\vec{\mathbf{u}} = (\delta\vec{\mathbf{u}}_1,\delta\vec{\mathbf{u}}_2)\in V \mid \delta\vec{\mathbf{u}}_1 = \vec{\mathbf{u}}_{1,D} \text{ on } \Gamma_{1,D} \text{ and } \delta\vec{\mathbf{u}}_2 = \vec{\mathbf{u}}_{2,D} \text{ on } \Gamma_{2,D} \mid [\![\delta\vec{\mathbf{u}}\cdot\vec{\mathbf{n}}]\!]-g\leq 0 \text{ on } \Gamma_{1,C}\}.$$

We assume that $\vec{\mathbf{f}} = (\vec{\mathbf{f}}_1,\vec{\mathbf{f}}_2)$ belongs to $L^2(\Omega_1)^d\times L^2(\Omega_2)^d$, $\vec{\ell} = (\vec{\ell}_1,\vec{\ell}_2)$ belongs to $L^2(\Gamma_{1,N})^d\times L^2(\Gamma_{2,N})^d$, and $\vec{\mathbf{u}}_D$ belongs to $H^{\frac{3}{2}}(\Gamma_{1,D})^d\times H^{\frac{3}{2}}(\Gamma_{2,D})^d$.

Definition 2.8 *We introduce a primal-mixed formulation of the elasto-plastic two-body contact problem and project the stresses onto the admissible set by the* plastic projector *:*

$$
\mathcal{P}(\tau) := \begin{cases}
\tau & \text{if } |\tau^{dev}|_F \leq \sigma^0_\alpha\\
\Big(\dfrac{\xi_{\alpha,iso}}{2G_\alpha+\xi_{\alpha,iso}}\Big)+\Big(1-\dfrac{\xi_{\alpha,iso}}{2G_\alpha+\xi_{\alpha,iso}}\Big)\dfrac{\sigma^0_\alpha}{|\tau^{dev}|_F}\tau^{dev}+\dfrac{1}{d}tr(\tau)\mathbb{I}_{d\times d} & \text{if } |\tau^{dev}|_F \geq \sigma^0_\alpha
\end{cases} \tag{7}
$$

with the shear modulus G_α of the α-th body material.

We consider a test function[1] $\delta\vec{\mathbf{u}}$ in the space of all (smooth) admissible variations of $\vec{\mathbf{u}}$ satisfying possibly homogeneous Dirichlet/Newmann conditions on the appropriate part of $\partial\Omega$.

Definition 2.9 *We define the semi-linear form $\mathcal{W}^{int}(.,.)$, called the internal virtual work (associated to the internal energy $\mathcal{E}^{int}$), the linear form $\mathcal{W}^{ext}(.)$, called the external virtual work, and we denote by $\mathcal{W}^{cb}(.,.)$ the contact (including Dirichlet boundary conditions) virtual work :*

$\boldsymbol{W}^{int}(\vec{\mathbf{u}},\delta\vec{\mathbf{u}}) := \sum_{\alpha=1,2}\int_{\Omega_\alpha}\boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha):\boldsymbol{\varepsilon}(\delta\vec{\mathbf{u}}_\alpha)\,d\Omega = \sum_{\alpha=1,2}\int_{\Omega_\alpha}\mathcal{P}(\mathbf{H}_\alpha^{-1}\boldsymbol{\varepsilon}(\vec{\mathbf{u}}_\alpha)):\boldsymbol{\varepsilon}(\delta\vec{\mathbf{u}}_\alpha)\,d\Omega = \delta\mathcal{E}^{int}(\vec{\mathbf{u}}) = \partial_{\vec{\mathbf{u}}}\mathcal{E}^{int}(\delta\vec{\mathbf{u}})$

$\boldsymbol{W}^{ext}(\delta\vec{\mathbf{u}}) := \sum_{\alpha=1,2}\int_{\Omega_\alpha}\vec{\mathbf{f}}_i\delta\vec{\mathbf{u}}_\alpha\,d\Omega + \sum_{\alpha=1,2}\int_{\Gamma_{\alpha,N}}\vec{\ell}_\alpha\delta\vec{\mathbf{u}}_\alpha\,d\Gamma$

$\mathcal{W}^{cb} := \mathcal{W}^c + \mathcal{W}^b$ *with* $\mathcal{W}^c(\vec{\mathbf{u}},\delta\vec{\mathbf{u}}) = \int_{\Gamma_{1,C}}\boldsymbol{\sigma}_n(\vec{\mathbf{u}}_1)[\![\delta\vec{\mathbf{u}}\cdot\vec{\mathbf{n}}]\!]\,d\Gamma.$

Following [3] and [8], one can derive for the general elasto-plastic case by replacing the contact and frictional constraints, leading to a mixed formulation of the form:

[1] or a virtual displacement $\vec{\mathbf{u}}^* = \vec{\mathbf{u}}+\delta\vec{\mathbf{u}}$, with operator δ verifying the following properties:

$$\delta\delta\vec{\mathbf{u}} = 0,\quad \delta(\nabla\vec{\mathbf{u}}) = \nabla\delta\vec{\mathbf{u}},\quad \delta\int_\Omega \vec{\mathbf{u}}d\Omega = \int_\Omega\delta\vec{\mathbf{u}}d\Omega,\quad \delta\vec{\mathbf{u}} = 0 \text{ on } \Gamma_D.$$

$$\begin{cases} \text{Find } (\vec{\mathbf{u}}, \vec{\boldsymbol{\lambda}}_n, \vec{\boldsymbol{\lambda}}_t) \in V \times \Lambda_n \times \Lambda_t \text{ such that} \\ \mathcal{W}^{int}(\vec{\mathbf{u}}, \delta\vec{\mathbf{u}}) + \mathcal{B}_n(\vec{\boldsymbol{\lambda}}_n, \delta\vec{\mathbf{u}}) + \mathcal{B}_t(\vec{\boldsymbol{\lambda}}_t, \delta\vec{\mathbf{u}}) \geq \mathcal{W}^{ext}(\delta\vec{\mathbf{u}} - \vec{\mathbf{u}}) \qquad \forall \delta\vec{\mathbf{u}} \in V \\ \mathcal{B}_n(\vec{\boldsymbol{\mu}}_n \vec{\boldsymbol{\lambda}}_n, \vec{\mathbf{u}}) + \mathcal{B}_t(\vec{\boldsymbol{\mu}}_n \vec{\boldsymbol{\lambda}}_n, \vec{\mathbf{u}}) \geq \langle \vec{\boldsymbol{\mu}}_n \vec{\boldsymbol{\lambda}}_n, g \rangle_{\Lambda_n} \qquad \forall (\vec{\boldsymbol{\mu}}_n, \vec{\boldsymbol{\mu}}_t) \in \Lambda_n \times \Lambda_t. \end{cases} \tag{8}$$

Here, the dual normal cone and the frinctional tangential cone are defined as follows::

$$\Lambda_n := \{\mu \in H^{1/2}(\Gamma_{1C}) \;/\; \mu \leq 0 a.e.\}' \quad \text{and } \Lambda_t = \{\mu \in H^{1/2}(\Gamma_{1C}) \;/\; \|\mu\|_{H^{1/2}} \leq F \; a.e.\}$$

where $\langle .,. \rangle_{\Lambda_n}$ denotes the dual pairing on Λ_n. The bilinear forms that incorporate the contact and frictional conditions are defined by $\mathrm{B}_n : \Lambda_n \times V \longrightarrow \mathbb{R}, \qquad \mathcal{B}_n(\vec{\boldsymbol{\mu}}, \vec{\mathbf{w}}) := \langle \vec{\boldsymbol{\mu}}, [\![\vec{\mathbf{w}}]\!]_n \rangle_{\Lambda_n}$,

$\mathrm{B}_t : \Lambda_t \times V \longrightarrow \mathbb{R}, \qquad \mathcal{B}_t(\vec{\boldsymbol{\mu}}, \vec{\mathbf{w}}) := \langle \vec{\boldsymbol{\mu}}, F[\![\vec{\mathbf{w}}]\!]_t \rangle_{L^2(\Gamma_{1C})}$.

3 Nitsche's Method Formulations

Augmented Lagrangians and Lagrangians are constrained optimization tools that were naturally applied by Rockafellar (1974-1976) to contact problems involving deformable solids. The augmented Lagrangian method has since become widely established for the approximation and resolution of contact problems in both small and large strains, mainly following the research of Curnier and Alart (1988-1991) and Simo and Laursen (1992). The method proposed by Nitsche (1971) was initially designed to allow for a Dirichlet-type boundary condition to be weakly enforced, specifically avoiding the use of Lagrange multipliers. It has only recently been extended to contact conditions, with or without friction, by Chouly and Hild (2013). The close connection between Nitsche's method and Lagrangian methods is quite clear, and the objective of [9] is to shed some light on this relationship. In this section, assuming that both the solution $\vec{\mathbf{u}}$ and the test functions $\delta\vec{\mathbf{u}}$ are sufficiently regular, we derive from the equilibrium equations and Green's formula:

$\mathbf{W}^{int}(\vec{\mathbf{u}}, \delta\vec{\mathbf{u}}) - \sum_{\alpha=1,2} \int_{\Gamma_{\alpha,D}} \boldsymbol{\sigma}_n(\vec{\mathbf{u}}\alpha)\vec{\mathbf{n}}_\alpha \cdot \delta\vec{\mathbf{u}}_\alpha \, d\Gamma - \int_{\Gamma_{1,C}} \boldsymbol{\sigma}_n(\vec{\mathbf{u}}_1)[\![\delta\vec{\mathbf{u}} \cdot \vec{\mathbf{n}}]\!] \, d\Gamma = \mathcal{W}^{ext}(\delta\vec{\mathbf{u}})$.

3.1 Nitsche's Formulation of General Elasto-Plastic Constitutive Law with Friction

Let $\theta \in \mathbb{R}$ be a fixed parameter used to recover different variants of the Nitsche method, as in the linear elastic setting (see article [5]). With the splitting $[\![\vec{\mathbf{u}} \cdot \vec{\mathbf{n}}]\!] = ([\![\vec{\mathbf{u}} \cdot \vec{\mathbf{n}}]\!] - \theta\gamma\mathcal{D}\sigma_n(\vec{\mathbf{u}})(\delta u)) + \theta\gamma\mathcal{D}\sigma_n(\vec{\mathbf{u}})(\delta u))$,

As presented in [6] for Tresca friction, we could reformulate the Coulomb friction condition using the projection $\mathbf{P}_{\mathbb{B}(0,\tau)}$. In fact, for a given positive function γ, the friction condition is equivalent to the non-smooth equation:

$$\boldsymbol{\sigma}_{\vec{\mathbf{t}}}(\vec{\mathbf{u}}) = P_{B(-F\boldsymbol{\sigma}_n(\vec{\mathbf{u}}))}(\boldsymbol{\sigma}_{\vec{\mathbf{t}}}(\vec{\mathbf{u}}) - \gamma\dot{\vec{\mathbf{u}}}). \tag{9}$$

To simplify notations, let use denote by $\mathbf{P}_{n,F}$ the map corresponding to a pair of projections:

$$\mathbf{P}_{n,F}(\vec{\mathbf{x}}) = -(\vec{\mathbf{x}} \cdot \vec{\mathbf{n}})_- \vec{\mathbf{n}} + \mathbf{P}_{\mathbb{B}(0,F(\vec{\mathbf{X}} \cdot \vec{\mathbf{n}})_-)}(\vec{\mathbf{x}} - (\vec{\mathbf{X}} \cdot \vec{\mathbf{n}})\vec{\mathbf{n}}). \tag{10}$$

This application projects the normal part of x onto $\mathbb{R}_-$ and the tangential part onto the ball $\mathbb{B}$ of center 0 and radius $F(\vec{\mathbf{X}} \cdot \vec{\mathbf{n}})_-$, where F is the friction coefficient.

As a result, contact and friction conditions, in the case of projection, are formulated as:

$$\sigma_n(\vec{\mathbf{u}}) = \mathbf{P}_{n,F}(\sigma_n(u) - \frac{Qu}{\gamma} + \frac{g}{\gamma}n + \frac{qw_T}{\gamma}). \tag{11}$$

and Q is the $ndim \times ndim$ matrix

$$Q := \alpha I_{ndim} + (1-\alpha)\vec{\mathbf{n}}\vec{\mathbf{n}}^\top$$

We shall consider that the sliding velocity is approximated by $q(u_T - w_T)$, where the expression of q and w_T depend on the time integration scheme used. Using this operator, we obtain the equilibrium equation :

$$
\begin{cases}
\mathcal{W}^{int}(\vec{\mathbf{u}},\delta\vec{\mathbf{u}}) - \sum_{\alpha=1,2}\int_{\Gamma_{\alpha,D}} \boldsymbol{\sigma}_n(\vec{\mathbf{u}}_\alpha)\vec{\mathbf{n}}_\alpha\cdot\delta\vec{\mathbf{u}}_\alpha \,\mathrm{d}\Gamma - \int_{\Gamma_{C1}} \theta\gamma\sigma_n(u_1)\cdot\mathcal{D}\sigma_n(u_1)[v]d\Gamma \\
+\int_{\Gamma_{C1}} \gamma\mathbf{P}_{n,F}(\sigma_n(u_1) - \frac{Qu}{\gamma} + \frac{g}{\gamma}n + \frac{qw_T}{\gamma})\cdot(\theta\mathcal{D}\sigma_n(u_1)[v] - \frac{v}{\gamma})d\Gamma = \mathcal{W}^{ext}(\delta\vec{\mathbf{u}}).
\end{cases}
\tag{12}
$$

Remark 3.1 *Note that, for the numerical solving, when $\theta \neq 0$, the tangent system involves the second-order derivative: $\mathcal{D}^2\boldsymbol{\sigma}(\vec{\mathbf{u}})[\delta\vec{\mathbf{u}},\overset{*}{\vec{\mathbf{u}}}]$, which can be very complex for non-classical constitutive laws. This emphasizes the interest in the non-symmetric variant $\theta = 0$, for which the method is simpler. As in the small strain case [4], the interest of the symmetric variant $\theta = 1$ lies mostly in its derivation from a potential (see [2], [5]) and the symmetry of the tangent problem, while the interest in the skew-symmetric variant $\theta = -1$ lies in its robustness with respect to the Nitsche parameter γ (see [3]).*

Using this, Nitsche's contact term, *corresponding to the virtual contact work for contact with friction*, reads as :

$$
\mathcal{W}^c = -\int_{\Gamma_{C1}} \theta\gamma\sigma_n(u_1)\cdot\mathcal{D}\sigma_n(u_1)[v]d\Gamma + \int_{\Gamma_{C1}} \gamma\mathbf{P}_{n,F}(\sigma_n(u_1) - \frac{Qu}{\gamma} + \frac{g}{\gamma}n + \frac{qw_T}{\gamma})\cdot(\theta\mathcal{D}\sigma_n(u_1)[v] - \frac{v}{\gamma})d\Gamma.
$$

Remark 3.2 *In [8], the Nitsche-based finite element method for contact with Coulomb friction, considering both static and dynamic elastic situations, is studed. In the dynamic case, existence and uniqueness of the space semi-discrete problem are guaranteed for every value of the friction coefficient and the Nitsche parameter. In the static case, if the Nitsche parameter is is sufficiently large, existence is ensured for any friction coefficient, and uniqueness can be obtained provided that the friction coefficient is below a bound dependent on the mesh size.*

Remark 3.3 *From [9], we understand that Penalty methods, which replace the set of inequalities associated with contact with a non-linear inequality approximating them, remain primal and are easy to implement. However, consistency is compromised as a small amount of penetration, controlled by the penalty parameter, is allowed. Additionally, the selection of the penalty parameter requires careful consideration. Indeed, as the penalty parameter is decreased to enhance the approximation of contact conditions, the discrete problem becomes stiffer and ill-conditioned, potentially causing iterative solvers such as semi-smooth Newton methods to fail to converge. It is observed that the non-penetration condition is more effectively respected with the Nitsche method. Furthermore, it is noted that the stress approximation is affected by spurious oscillations in frictional problems, which are more pronounced in the case of the penalty method.*

Remark 3.4 *The parameter γ, appearing as a penalization parameter in the penalized formulation, as an augmentation parameter in the augmented Lagrangian formulation, and as a Nitsche parameter in the formulation in (12), plays a rather similar role across all three approaches. While the numerical solution is relatively unaffected by the value of γ in both the augmented Lagrangian and Nitsche formulations, a minimum value must be maintained in the Nitsche method to preserve problem coercivity. For the penalty method, a balance must be struck between a large value of γ that ensures a good approximation of the contact and friction conditions, and a moderate value that does not hinder the convergence of the Newton method. Although the concerns differ across the three approaches, the optimal values of γ are similar. A priori error analyses, within the framework of small strains for both the Nitsche method [5] and the first-order convergence of the penalty method [7], indicate a dependency on $\gamma = \gamma_0/h$, where h is the mesh size. Furthermore, the study by [17] and the dimensional analysis conducted in [15] in the context of the augmented Lagrangian conclude that γ_0 has the dimension of an elastic modulus. It is thus quite natural to choose $\gamma = K/h$ where $K = \lambda + \frac{2}{3}\mu$ represents the bulk modulus (with λ and μ denoting the Lamé coefficients). Notably, for the Nitsche method applied to large strains, this value may not be sufficient to ensure coercivity. Indeed, when deformations are significant, the value of γ_0 should rather be linked to the maximum value of the tangent moduli of elasticity.*

4 Numerical Results

We conclude this note with numerical results aimed at testing different models, utilizing classical finite element discretization of the previously discussed Nitsche method, implemented in SYSTUS/SYSWELD.

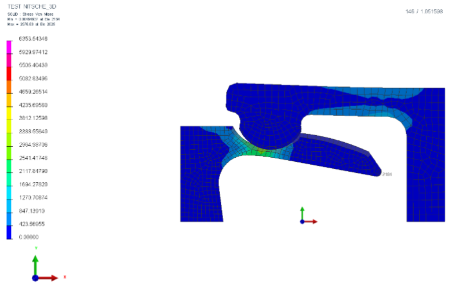

Figure 1: Clip details : elastoplastic contact

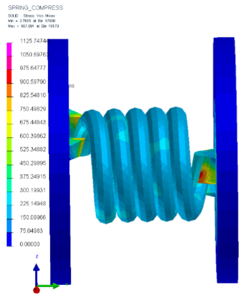

Figure 2: spring : elastoplastic multi-auto-contacts

In this work, we employ *segment-to-segment integration on each slave element and enforce contact constraints at each slave element*. Following the approach in [15], all integrals related to contact are evaluated using *Legendre-Gauss quadrature* on the faces of the mesh elements that comprise the slave surface. The choice of Gauss points is advantageous as they outnumber the nodes per element, and calculations are inherently performed at these points. To facilitate contact analysis, a pair is formed by a point on a slave (or master, in cases of self-contact) surface and a corresponding projected point on the nearest master surface (or rigid obstacle). Contact detection then proceeds independently at each quadrature point, allowing for the possibility that different quadrature points on the same slave surface element face may correspond to points on various faces of the master surface. For efficient identification of candidate master element faces opposite a given quadrature point, we utilize a simple heuristic method involving R-tree organized influence boxes, achieving logarithmic computational complexity. The user documentation of GetFEM, as referenced in [14], provides additional details on the heuristic criteria applied for contact detection.

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